NAME (print): KEY

Math 203 Spring 2013—Exam 4

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Work carefully and neatly and remember that I cannot grade what I cannot read. You must show all relevant work in the appropriate space. You may receive no credit for a correct answer if there is insufficient supporting work. Notes, books and programable calculators are NOT ALLOWED.

[15pt] 1. Fill in the blanks with *A(lways), S(ometimes), N(ever)* so that the following are correct statements.

(a) If A is an $n \times n$ matrix and u is a nonzero vector in $\mathbb{R}^n$, such that $Au = 5u$, then 5 is __A__ an eigenvalue of A.

(b) If 5 is an eigenvalue of A and if B is similar to A, then 5 is __A__ an eigenvalue of B.

(c) If A is a 3×3 matrix and 3 and 5 are the only eigenvalues of A, then A is __S__ diagonalizable.

(d) If $\text{char}(A) = (5-x)^2(3-x)$, and the dimension of the eigenspace of 5 is one, then A is __N__ diagonalizable.

(e) Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. Let A be the matrix of T with respect to the standard matrix and suppose that B is a matrix that is similar to A. Then B __A__ represents the matrix of T with respect to some basis of $\mathbb{R}^n$.

[15] 2. Find the characteristic polynomial, eigenvalues and a basis for each corresponding eigenspace for the matrix below:

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\det\begin{pmatrix} 1-\lambda & -1 \\ -1 & 1-\lambda \end{pmatrix} = (1-\lambda)^2 - 1 = \lambda^2 - 2\lambda + 1 - 1$$

$$= \boxed{\lambda^2 - 2\lambda} = \lambda(\lambda - 2)$$

E.V $\lambda = 0, \lambda = 2$

$$A - 0I = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \quad \boxed{\begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ for } \lambda = 0}$$

$$A - 2I = \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix} \sim \begin{pmatrix} -1 & -1 \\ 0 & 0 \end{pmatrix} \quad \boxed{\begin{pmatrix} 1 \\ -1 \end{pmatrix} \text{ for } \lambda = 2}$$

[12] 3. Let $P = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$, (so $P^{-1} = \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}$) and let $D = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$. If $A = PDP^{-1}$, compute A^4.

$$A^4 = PD^4P^{-1} = \underbrace{\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 16 & 0 \\ 0 & 1 \end{pmatrix}}\begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 16 & 2 \\ 16 & 1 \end{pmatrix}\begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} -14 & 30 \\ -15 & 31 \end{pmatrix}$$

[20] 4. Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -2 \\ 0 & 4 & -3 \end{bmatrix}$ (then A is diagonalizable).

(a) Find the eigenvalues of A and then find a diagonal matrix similar to A.

$$\text{char}(A) = \det(A - \lambda I) = (1-\lambda)\det\begin{pmatrix} 3-\lambda & -2 \\ 4 & -3-\lambda \end{pmatrix}$$

$$= (1-\lambda)(3-\lambda)(-3-\lambda) - (-2)(4)$$

$$= (1-\lambda)(\lambda^2 - 9 + 8)$$

$$= (1-\lambda)(\lambda^2 - 1)$$

$\boxed{\lambda = 1, -1}$

$\boxed{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}}$

(b) Find an invertible matrix P such that $A = PDP^{-1}$, where D is the matrix you found in part (a). Find eigenvectors

$$A - 1I = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & -2 \\ 0 & 4 & -4 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

Two free variables

$$A + 2I = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & -2 \\ 0 & 4 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 \\ 1/2 \\ 1 \end{pmatrix}$$

P

$$\boxed{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1/2 \\ 0 & 1 & 1 \end{pmatrix}}$$

[10] 5. Let $T : \mathbb{P}_2 \to \mathbb{R}^3$ be defined by $T(p) = (2p(1), p(-1), p(3))$. Then T is a linear transformation (you do not have to prove this). Find the matrix of T relative to the bases $\{t^2, t, 1\}$ (note the order) and the standard basis of $\mathbb{R}^3$.

$T(t^2) = (2, 1, 9)$

$T(t) = (2, -1, 3)$

$T(1) = (2, 1, 1)$

$$\boxed{A = \begin{pmatrix} 2 & 2 & 2 \\ 1 & -1 & 1 \\ 9 & 3 & 1 \end{pmatrix}}$$

[10] 6. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation given by $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5x + y \\ 2x + 4y \end{pmatrix}$. Let $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right\}$ (so $\mathcal{B}$ is a basis of $\mathbb{R}^2$). Find the $\mathcal{B}$-matrix for T.

$T\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 6 \end{pmatrix} = 6\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$T\begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 3 \\ -6 \end{pmatrix} = 3\begin{pmatrix} 1 \\ -2 \end{pmatrix}$

So $[T]_{\mathcal{B}} = \begin{pmatrix} 6 & 0 \\ 0 & 3 \end{pmatrix}$

[10] 7. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation, such that the matrix of T with respect to the $\mathcal{B} = \{b_1, b_2, b_3\}$ basis is

$$A = \begin{pmatrix} 4 & -2 & 3 \\ 1 & 5 & 0 \\ 0 & -1 & 3 \end{pmatrix}.$$ What is $[T(b_1 + 2b_2 - b_3)]_{\mathcal{B}}$?

$\begin{pmatrix} 4 & -2 & 3 \\ 1 & 5 & 0 \\ 0 & -1 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 4-4-3 \\ 1+10 \\ -2-3 \end{pmatrix} = \begin{pmatrix} -3 \\ 11 \\ -5 \end{pmatrix}$

[8] 8. Let $u = \begin{pmatrix} 2 \\ -5 \\ -1 \end{pmatrix}$ and $w = \begin{pmatrix} -7 \\ -4 \\ 6 \end{pmatrix}$. Compute $u \cdot w$ and $\|u\|^2$.

$u \cdot w = -14 + 20 - 6 = 0$

$\|u\|^2 = 4 + 25 + 1 = 30$